

ON ARITHMETIC PROGRESSIONS ON GENUS TWO CURVES

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ABSTRACT. We study arithmetic progression in the x -coordinate of rational points on genus two curves. As we know, there are two models for the curve C of genus two: $C : y^2 = f_5(x)$ or $C : y^2 = f_6(x)$, where $f_5, f_6 \in \mathbb{Q}[x]$, $\deg f_5 = 5$, $\deg f_6 = 6$ and the polynomials f_5, f_6 do not have multiple roots. First we prove that there exists an infinite family of curves of the form $y^2 = f(x)$, where $f \in \mathbb{Q}[x]$ and $\deg f = 5$ each containing 11 points in arithmetic progression. We also present an example of $F \in \mathbb{Q}[x]$ with $\deg F = 5$ such that on the curve $y^2 = F(x)$ twelve points lie in arithmetic progression. Next, we show that there exist infinitely many curves of the form $y^2 = g(x)$ where $g \in \mathbb{Q}[x]$ and $\deg g = 6$, each containing 16 points in arithmetic progression. Moreover, we present two examples of curves in this form with 18 points in arithmetic progression.

1. INTRODUCTION

Let $f \in \mathbb{Q}[X]$ be a polynomial without multiple roots and let us consider the curve $C : y^2 = f(x)$. We say that rational points $P_i = (x_i, y_i)$ for $i = 1, 2, \dots, n$ are in *arithmetic progression* on the curve C , if rational numbers x_i are in arithmetic progression for $i = 1, 2, \dots, n$. A positive integer n will be called *the length of arithmetic progression* on the curve C . A natural question arises here: How long can arithmetic progression on the curve $y^2 = f(x)$ with fixed degree of f be? Through the whole paper by a point we mean a rational one.

In case of polynomials of degree one, this question is equivalent to the question about the number of squares which form an arithmetic progression.

It is not difficult to show that there exists an infinite family \mathcal{A}_1 of polynomials of degree one, with the property that for each $f \in \mathcal{A}_1$ there are 3 points in arithmetic progression on the curve $y^2 = f(x)$ (of genus 0). It turns out, however, which was already proved by Fermat, that it is impossible to construct arithmetic progression composed of four squares.

In paper [1] Allison has shown that there exists an infinite family \mathcal{A}_2 of polynomials of degree two, such that for each $f \in \mathcal{A}_2$ on the curve $y^2 = f(x)$ (of genus 0) eight points lie in arithmetic progression.

In the case of polynomials of degree three, Bremner in [2] has constructed an infinite family \mathcal{A}_3 with such a property that for every $f \in \mathcal{A}_3$ on the curve $y^2 = f(x)$ (of genus 1) 8 points lie in arithmetic progression. A similar result with the use of other methods was obtained by Campbell in [3].

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In the case of polynomials of degree four, in [8] we have constructed an infinite family \mathcal{A}_4 with such property that for every $f \in \mathcal{A}_4$ there are 12 points in arithmetic progression on the curve $y^2 = f(x)$ (of genus 1).

It is worth noting that MacLeod in [5] has constructed polynomials F_i , ($i = 1, 2, 3, 4$) of degree four such that on each curve $y^2 = F_i(x)$ there are 14 points in arithmetic progression.

In all above cases, each of the families \mathcal{A}_2 , \mathcal{A}_3 , \mathcal{A}_4 is parametrized by rational points on some elliptic curve of positive rank.

It is reasonable to define the following quantities

$$m(d) := \max\{k : \text{there exists a polynomial } g \in \mathbb{Q}[x] \text{ of degree } \deg g = d \text{ such that on the curve } y^2 = g(x) \text{ there are } k \text{ points in arithmetic progression}\},$$

$$M(d) := \max\{k : \text{there exists an infinite family } \mathcal{A}_d \text{ of polynomials of degree } d, \text{ such that for every } g \in \mathcal{A}_d \text{ there are } k \text{ points in arithmetic progression on the curve } y^2 = g(x) \}.$$

We have obvious inequality $m(d) \geq M(d)$. The above results can be grouped in the following manner:

d	1	2	3	4
$m(d)$	3	≥ 8	≥ 8	≥ 14
$M(d)$	3	≥ 8	≥ 8	≥ 12

TABLE 1

In this paper we will concentrate on quantities $m(d)$ and $M(d)$ for $d = 5, 6$. Let us note that it corresponds to the construction of arithmetic progressions on hyperelliptic curves of genus 2. In case of $d = 5$ we show that $m(5) \geq 12$ and $M(5) \geq 11$. When $d = 6$, we first show that there exists a polynomial $G(t, x) \in \mathbb{Q}(t)[x]$ such that there are 14 points in arithmetic progression on the curve $y^2 = G(t, x)$. Using another approach we prove that $m(6) \geq 18$ and $M(6) \geq 16$.

2. CASE OF $d = 5$

Using a method similar to that used by Campbell in [3] we will show the following

Theorem 2.1. *There exist polynomials $F_i(t, x) \in \mathbb{Q}(t)[x]$, ($i = 1, 2$) of degree $\deg_x F_i = 5$ such that on the curve $y^2 = F_i(t, x)$ eleven $\mathbb{Q}(t)$ -rational points lie in arithmetic progression.*

Proof. Let u be a variable and let us consider a polynomial

$$g(u, x) = (x - u)^2 \prod_{i=1}^{10} (x - i).$$

As we know, there is exactly one pair of polynomials $h, f \in \mathbb{Q}(u)[x]$ such that $\deg_x h = 6$, $\deg_x f = 5$ and

$$g(u, x) = h(u, x)^2 - \frac{25}{1048576} f(u, x).$$

In our case the polynomial f is in the form of

$$f(u, x) = a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0,$$

where

$$\begin{aligned} a_0 &= 5695244944u^2 - 12894461800u + 263250625, \\ a_1 &= -8(533634200u^2 - 873304794u - 1611807725), \\ a_2 &= 32(37257330u^2 - 8034400u - 396302603), \\ a_3 &= -3520(41536u^2 + 145226u - 1285845), \\ a_4 &= 3520(1888u^2 + 31152u - 193477), \\ a_5 &= -(6645760u - 36551680). \end{aligned}$$

If u is rational and $u \neq 11/2$ then the polynomial $f(u, x)$ is without multiple roots. Thus, we see that in this case there are 10 points in arithmetic progression on the curve $y^2 = f(u, x)$.

Let us now consider curve Q_1 with the equation $p_1^2 = f(u_1, 11)$. This is a quadric with rational point $(11, 16225)$. Using the standard method we have a parametrization of the curve Q_1 given by:

$$\begin{aligned} p_1(t) &= \frac{16225(5695244944t^2 - 794728t + 1)}{5695244944t^2 - 1}, \\ u_1(t) &= \frac{11(4523021144t^2 + 2950t - 1)}{5695244944t^2 - 1}. \end{aligned}$$

If we now define $F_1(t, x) = f(u_1(t), x)$, where $u_1(t)$ is as above, then on the curve

$$C_1 : y^2 = F_1(t, x)$$

there are 11 points in arithmetic progression.

We can similarly parametrize the quadric Q_2 given by the equation $p_2^2 = f(u_2, 0)$ with rational point $(0, 16225)$. In this case the parametrization takes the form

$$\begin{aligned} p_2(t) &= \frac{16225(5695244944t^2 + 794728t + 1)}{5695244944t^2 - 1}, \\ u_2(t) &= \frac{32450t(397364t + 1)}{5695244944t^2 - 1}. \end{aligned}$$

If we now define $F_2(t, x) = f(u_2(t), x)$, where $u_2(t)$ is as above, then on the curve

$$C_2 : y^2 = F_2(t, x)$$

there are 11 points in arithmetic progression. \square

Finding of such a rational t that on the curve C_1 there are twelve points in arithmetic progression requires finding rational points on the curve

$$\begin{aligned} y^2 &= -543542815457978537904123051776t^4 - 5858532530788995918150400t^3 \\ &\quad + 611541611111856733408t^2 - 633115875308400t - 30556659591, \end{aligned}$$

or on the curve

$$\begin{aligned} y^2 &= 10452723797211797241575306232064t^4 + 16006824835104105921670400t^3 \\ &\quad - 4781502606421467214112t^2 - 3647410080111600t + 547548809049. \end{aligned}$$

Then, points with x -coordinates in $\{1, 2, \dots, 12\}$ (respectively with x -coordinates in $\{0, 1, \dots, 11\}$) will be in arithmetic progression on the curve C_1 . In the case

of the curve C_2 we obtain the same curves. It is easy to see that the above curves have \mathbb{Q}_p -rational points for every p , but unfortunately we did not manage to find a rational point on any of the above curves. It seems that finding a rational point on any of the curves (or showing that such points do not exist) may be a difficult task.

Remark 2.2. The statement of Theorem 2.1 can also be obtained using the following reasoning. Let us consider the polynomial

$$f(t, x) = (x - t) \prod_{i=1}^{11} (x - i).$$

Then there exist polynomials $p, F \in \mathbb{Q}(t)[x]$, such that $\deg_x p = 6, \deg_x F = 5$ and

$$f(t, x) = p(t, x)^2 - F(t, x)$$

Then the curve $C : y^2 = F(t, x)$ contains 11 points in arithmetic progression. Unfortunately, in this case the polynomials $F(t, 0)$ and $F(t, 12)$ are irreducible of degree 6, and each of the curves $y^2 = F(t, 0)$, $y^2 = F(t, 12)$ contain only finitely many rational points. Therefore, the curve C cannot be used to construct an infinite family of curves with the required property.

The following example found with the use of computer shows that $m(5) \geq 12$. Consider the curve

$$C : y^2 = 12x^5 - 322x^4 + 3208x^3 - 14438x^2 + 27980x - 16079.$$

We have the following points in arithmetic progression on the curve C :

$$\{(1, 19), (2, 55), (3, 37), (4, 1), (5, 11), (6, 31), (7, 35), \\ (8, 23), (9, 29), (10, 89), (11, 181), (12, 305)\}.$$

3. CASE OF $d = 6$

Let us begin with the following

Theorem 3.1. *There exists a polynomial $H(t, x) \in \mathbb{Q}(t)[x]$ of degree $\deg_x H = 6$, such that fourteen $\mathbb{Q}(t)$ -rational points lie in arithmetic progression on the curve $y^2 = H(t, x)$.*

Proof. Let t be a variable and let us consider a polynomial

$$h(t, x) = (x^2 - 15x + 4t) \prod_{i=1}^{14} (x - i).$$

Then there is exactly one pair of polynomials $g, H \in \mathbb{Q}(t)[x]$ such that $\deg_x g = 6, \deg_x H = 6$ and

$$h(t, x) = g(t, x)^2 - H(t, x).$$

In our case the polynomial H is in the form

$$H(t, x) = a_3(x(15 - x))^3 + a_2(x(15 - x))^2 + a_1(x(15 - x)) + a_0,$$

where

$$\begin{aligned}
a_0 &= 46228440064 - 37262033920t + 10620980224t^2 - 1420209280t^3 \\
&\quad + 106891216t^4 - 4876960t^5 + 144760t^6 - 2800t^7 + 25t^8, \\
a_1 &= 4(-790888960 + 642389312t - 177526160t^2 + 21803240t^3 \\
&\quad - 1364540t^4 + 42826t^5 - 630t^6 + 5t^7), \\
a_2 &= 2(35503616 - 29056640t + 7910592t^2 - 929040t^3 + 52318t^4 \\
&\quad - 1260t^5 + 7t^6), \\
a_3 &= 2(-261120 + 215008t - 58040t^2 + 6636t^3 - 350t^4 + 7t^5).
\end{aligned}$$

Therefore, we see that on the curve

$$C: y^2 = H(t, x)$$

fourteen points lie in arithmetic progression. These points are of the form $P_i = (i, g(t, i))$ for $i = 1, 2, \dots, 14$. \square

The first part of the proof of Theorem 3.1 suggests considering polynomials which are invariant with respect to the change of variables $x \rightarrow 15 - x$. Let us, therefore, consider the polynomial

$$(3.1) \quad f(x) = b_3(x(x - 15))^3 + b_2(x(x - 15))^2 + b_1x(x - 15) + b_0,$$

where

$$\begin{aligned}
b_0 &= (6p^2 - 22q^2 + 27r^2 - 11s^2)/47520, \\
b_1 &= (159p^2 - 517q^2 + 567r^2 - 209s^2)/11880, \\
b_2 &= (5496p^2 - 14872q^2 + 14337r^2 - 4961s^2)/11880, \\
b_3 &= (156p^2 - 308q^2 + 273r^2 - 91s^2)/30.
\end{aligned}$$

For f defined in this way we have

$$f(1) = f(14) = p^2, \quad f(2) = f(13) = q^2, \quad f(3) = f(12) = r^2, \quad f(4) = f(11) = s^2.$$

Therefore, we see that in order to obtain on the curve $y^2 = f(x)$ an arithmetic progression of the length 14, it is necessary to investigate a system of equations

$$(3.2) \quad \begin{cases} f(5) = (-14p^2 + 77q^2 - 162r^2 + 154s^2)/55 = u^2 \\ f(6) = (-21p^2 + 110q^2 - 210r^2 + 154s^2)/33 = v^2 \\ f(7) = (-60p^2 + 308q^2 - 567r^2 + 385s^2)/66 = w^2 \end{cases}$$

Using a substitution $(p, q, r, s, u) = (a + u, b + u, c + u, d + u, u)$ we obtain a parametrization of solutions of the first equation of system (3.2)

$$\begin{aligned}
(p, q, r, s, u) &= (14a^2 - 154ab + 77b^2 + 324ac - 162c^2 - 308ad + 154d^2, \\
&\quad 14a^2 - 28ab + 77b^2 - 324bc + 162c^2 + 308bd - 154d^2, \\
&\quad -14a^2 + 77b^2 + 28ac - 154bc + 162c^2 - 308cd + 154d^2, \\
&\quad 14a^2 - 77b^2 + 162c^2 - 28ad + 154bd - 324cd + 154d^2, \\
&\quad -14a^2 + 77b^2 - 162c^2 + 154d^2).
\end{aligned}$$

Now let us set

$$(3.3) \quad (a, b, c, d) = (946A, 946, 11(15A + 71), 441A + 505).$$

For a, b, c, d defined in this way, we get a parametric solution of the system (3.2) given by

$$(p, q, r, s, u, v, w) = (181144A^2 + 85170A - 42585, 59140A^2 - 118280A - 164589, \\ 17230A^2 - 112505A - 128454, 52874A^2 + 102845A + 68010, \\ 59140A^2 + 122004A + 42585, 43984A^2 + 104070A + 75675, \\ 15790A^2 + 107955A + 99984)$$

For p, q, r, s defined above, the coefficients of the polynomial $g_A(x) = 36f(x)$ are

$$b_0 = 36(128941675300A^4 + 235814377620A^3 + 34730973441A^2 \\ - 216866857320A - 132565503600), \\ b_1 = 4(A - 1)(254A + 219)(354070194A^2 + 848446325A + 620203644), \\ b_2 = (A - 1)(254A + 219)(35708622A^2 + 96399845A + 73722213), \\ b_3 = 4(A - 1)(254A + 219)(72474A^2 + 210275A + 164709).$$

For $A \in \mathbb{Q} \setminus S$, where $S = \{-240/233, -219/254, 1, 475/2\}$, the polynomial g_A does not have multiple roots. From this we can conclude that for $A \in \mathbb{Q} \setminus S$ on the curve

$$C_A : y^2 = g_A(x)$$

fourteen points lie in arithmetic progression. Now, it is an easy task to prove the following

Theorem 3.2. *There exist infinitely many $A \in \mathbb{Q}$ such that on the curve $C_A : y^2 = g_A(x)$ there are 16 points in arithmetic progression.*

Proof. Let us set $x = 0$ and consider the curve

$$C : y^2 = b_0(A).$$

It is easy to see that on C we have rational point $P = (1, 1342374)$. As we know, the curve of the form $y^2 = f_4(x)$, where $\deg f_4 = 4$, with rational point is birationally equivalent to an elliptic curve with Weierstrass' equation [6]. Using APECS program [4] we obtain that C is birational with the curve

$$E : y^2 + xy + y = x^3 - x^2 + 21015110653x + 1214962664541571.$$

For the curve E we have

$$\text{Tors } E(\mathbb{Q}) = \{\mathcal{O}, (-51365, 25682)\},$$

and again using APECS we obtain that free part of $E(\mathbb{Q})$ is generated by

$$G_1 = (-45989, -12274606), G_2 = (751451, -664705966), \\ G_3 = (-17669, -28941646), G_4 = (24913585/256, 264676595567/4096).$$

As an immediate consequence, we get that there are infinitely many rational points on the curve C and all but finitely many define the curve $C_A : y^2 = g_A(x)$ with 16 points in arithmetic progression. \square

To show that $m(6) \geq 18$ we have taken the polynomial of the form

$$(3.4) \quad h(x) = c_3(x(x-19))^3 + c_2(x(x-19))^2 + c_1x(x-19) + c_0.$$

With help of computer we found the following numbers c_0, c_1, c_2, c_3 such that the polynomial (3.4) has values which are squares of integers for $x = 1, 2, \dots, 18$:

c_0	c_1	c_2	c_3
358043904	18892800	321792	1664
864002304	37085184	524544	2432

TABLE 2

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